**Perturbations:** Non-relativistic matter. Wave shorter than the distance to horizon

Fluid equations in comoving coordinates. We introduce the comoving coordinates:

\[ \vec{r} = a(t) \vec{x}(t) \]
\[ \vec{u} = \frac{d\vec{r}}{dt} = H \vec{r} + \vec{v} \]
\[ \vec{v} = a \frac{d\vec{x}}{dt} \]

Proper distance \( \vec{r} \), comoving distance \( \vec{x} \), proper velocity \( \vec{u} \), peculiar velocity \( \vec{v} \)

Hydrodynamic equations are written in standard form

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0 \]  continuity equation
\[ \frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla) \vec{u} = -\frac{1}{\rho} \nabla P - \nabla \Phi \]  Euler equation
\[ \nabla^2 \Phi = 4\pi G \rho \]  Poisson equation

Introduce peculiar gravitational potential:

\[ \Phi = \frac{2}{3} \pi G \rho_b r^2 + y \]

Now we need to change variables from proper to comoving. Spatial derivatives are simple to handle:

\[ \frac{\partial}{\partial \vec{r}} = \frac{1}{a} \frac{\partial}{\partial \vec{x}} \]

Time derivatives at constant \( \vec{r} \) should be changed to time derivates at constant \( \vec{x} \).
This gives two terms: derivative with respect to time at constant comoving coordinate and a term due to the fact that constant \( \vec{r} \) means that \( \vec{x} \) changes. So, we get physical property at different comoving coordinates: a gradient should be present:
Here is an example: density

In $\Delta t$ time interval density changes by:

$$\Delta \rho \Big|_r = \Delta \rho \Big|_x + \frac{\Delta \rho}{\Delta x} \Delta x$$

Here $\Delta x$ is defined by condition:

$$\vec{r} = \text{const} \Rightarrow \Delta a \cdot \vec{x} + a \Delta x \Rightarrow \Delta x = - \frac{\Delta a}{a} x$$

Now: $\Delta \rightarrow d$

$$\frac{\partial \rho}{\partial t} \Big|_r = \frac{\partial \rho}{\partial t} \Big|_x - \frac{d}{dt} \left( \vec{x} \nabla \vec{x} \right) \rho$$

In the same manner we get:

$$\frac{\partial \vec{u}}{\partial t} \Big|_r = \frac{\partial \vec{u}}{\partial t} \Big|_x - \frac{1}{a} \frac{d}{dt} \left( \vec{x} \nabla \vec{x} \right) \vec{u}$$

Because we had: 

$$\vec{u} = \dot{a} \vec{x} + \vec{v} \Rightarrow \frac{\partial \vec{u}}{\partial t} \Big|_x = \dot{a} \vec{x} + \frac{\partial \vec{v}}{\partial t}$$

Now we need to change variables from $\vec{u}$ to $\vec{v}$

$$\nabla_r (\rho \vec{u}) = \frac{1}{a} \nabla_x \left( \rho \dot{a} \vec{x} + \rho \vec{v} \right) = \frac{1}{a} \nabla_x (\rho \vec{v}) + \frac{\dot{a}}{a} \nabla_x (\rho \vec{x}) =$$

$$= \frac{1}{a} \nabla_x (\rho \vec{v}) + \frac{\dot{a}}{a} \vec{x} \nabla \rho + \frac{\dot{a}}{a} \rho \nabla \vec{x} =$$

$$\nabla^2 \vec{x} = 3 \quad \nabla_r \vec{u} = \frac{\dot{a}}{a} \vec{x} \nabla \vec{u} + \frac{\vec{v}}{a} \nabla \left( \vec{a} \vec{x} + \vec{v} \right) =$$

$$= \frac{\dot{a}}{a} \vec{x} \nabla \vec{u} + \frac{\vec{a}}{a} \vec{v} + \frac{\vec{v}}{a} \nabla \vec{v}$$

Put all the terms in the equations of hydrodynamics and omitting indexes $r$ in derivatives:

$$\frac{\partial \rho}{\partial t} + 3 \frac{\dot{a}}{a} \rho + \frac{1}{a} \nabla (\rho \vec{v}) = 0$$
Thus, the Euler equation gives:

\[ \frac{\partial \vec{v}}{\partial t} + \frac{\dot{a}}{a} \vec{v} + \frac{1}{a} \left( \vec{v} \nabla \right) \vec{v} = -\frac{1}{a} \frac{\nabla P}{\rho} - \frac{1}{a} \nabla \phi - \ddot{a} \vec{x} - \frac{\dot{a}}{a} \nabla x \left( \frac{2 \pi \rho_b r^2}{3} \right) \]

\[ - \left( \ddot{a} + \frac{4 \pi}{3} \frac{G \rho_b}{a} \right) \vec{x} = 0 \]

Now, let's handle the Poisson equation:

\[ \nabla^2 \Phi = \frac{1}{a^2} \nabla_x^2 \phi + \nabla_x^2 \left( \frac{2 \pi \rho_b r^2}{3} \right) = \frac{1}{a^2} \nabla_x^2 \phi + 4 \pi \frac{G \rho_b}{a^2} \]

Thus, the Poisson equation takes the form:

\[ \nabla^2 \phi = 4 \pi \frac{G}{a^2} (\rho - \rho_b) \]

Note that gravity effectively gets "stronger" with time because there is term \( a^2 \).

Only deviations from the homogeneity enter the right-hand-side ("source term")

If local density is smaller than the average density, perturbations in grav. Potential are negative. This acts as a negative mass.

Using the equations, we can re-cast them for equations of individual free particles:

\[ \frac{d \vec{v}}{d t} + \frac{\dot{a}}{a} \vec{v} = -\frac{1}{a} \nabla \phi \]

Introduce the momentum:

\[ \vec{p} = a \vec{v} = a^2 \vec{x} \Rightarrow \frac{d \vec{p}}{d t} = -\nabla \phi \]
Regime of small perturbations:

\[ \rho(x, t) = \rho_b(t)(1 + \delta(x, t)) \]

\[ \delta \ll 1 \]

We can neglect terms of higher order: \[ \delta^2, v^2, \ldots \]

Continuity equation:

\[
(1+\delta)\frac{\partial \rho_b}{\partial t} + \rho_b \frac{\partial \delta}{\partial t} + \frac{3}{a} (1+\delta) \dot{\rho}_b + \frac{\rho_b}{a} \nabla \cdot \vec{v} = 0
\]

This gives two equations:

\[
\frac{\partial \rho_b}{\partial t} + \frac{3}{a} \frac{\dot{a}}{a} \rho_b = 0
\]

\[
\delta + \frac{1}{a} \nabla \cdot \vec{v} = 0
\]

Euler equation:

\[
\frac{\partial \vec{v}}{\partial t} + \frac{\dot{a}}{a} \vec{v} = -\frac{\nabla \rho}{a \rho_b} - \frac{1}{a} \nabla \varphi
\]

Poisson equation:

\[
\nabla^2 \varphi = 4\pi G \rho_b \delta
\]

Take divergence of Euler eq:

\[
\frac{\partial \nabla \cdot \vec{v}}{\partial t} + \frac{\dot{a}}{a} \nabla \cdot \vec{v} = -\frac{\nabla^2 \rho}{a \rho_b} - 4\pi G \rho_b \delta
\]

From continuity eq we get:

\[
\nabla \cdot \vec{v} = -\dot{\delta}
\]

Combining these equations (get rid of div(v) and grav. Potential):

\[
\ddot{\delta} + 2 \frac{\dot{a}}{a} \dot{\delta} = \frac{\nabla^2 \rho}{a^2 \rho_b} + 4\pi G \rho_b \delta
\]

If \( \rho \) is negligible, we have a linear differential equation, which does not depend on the wavelength of the perturbation. In other words, in this regime all perturbations grow with the same rate.

\[
\ddot{\delta} + 2 \frac{\dot{a}}{a} \dot{\delta} = 4\pi G \rho_b \delta \quad , \quad \rho \approx 0
\]
Peculiar velocity and peculiar acceleration:

Equations:

\[ \nabla^2 \varphi = 4\pi G a^2 (\rho - \rho_b) \Rightarrow \varphi (\vec{x}) = -G a \int d^3 x' \frac{\rho (x') - \rho_b}{|x' - x|^b} \]

\[ \vec{g} (\vec{x}) = -\frac{\nabla \varphi}{a} = G a \int d^3 x' (\rho - \rho_b) \frac{\vec{x}' - \vec{x}}{|x' - x|^3} \]

Peculiar acceleration can be re-written in slightly different form:

\[ \vec{g} (\vec{x}) = G a \rho_b \int d^3 x' \delta (\vec{x}, t) \frac{\vec{x}' - \vec{x}}{|x' - x|^3} \]

Relation between \( g \) and \( v \) in the linear regime:

\[ \nabla \vec{g} = -4\pi G \rho_b a \delta \Rightarrow \delta = -\frac{\nabla \vec{g}}{4\pi G \rho_b a} \Rightarrow \dot{\delta} = -\frac{2}{a} \left( \frac{\nabla \vec{g}}{4\pi G \rho_b a} \right) \]

From continuity equation we get:

\[ \delta = -\frac{\nabla \vec{v}}{a} \]

Thus:

\[ \vec{v} = a \frac{\partial}{\partial t} \left( \frac{\vec{g}}{4\pi G \rho_b a} \right) + \frac{\vec{F} (\vec{x})}{a} \bigg|_{\nabla \vec{F} = 0} \]

\[ \Rightarrow \text{decaying mode} \]

For growing mode we find that velocity and acceleration are related:

\[ \vec{v} = a \frac{\partial}{\partial t} \left( \frac{\vec{g}}{4\pi G \rho_b a} \right) \Rightarrow \vec{v} \parallel \vec{g} \]

\[ \vec{g} = G \rho_b a \int d^3 x' \delta (x') \frac{\vec{x}' - \vec{x}}{|x' - x|^3} \Rightarrow \left( \frac{\vec{g}}{G \rho_b a} \right) = \int \left( \ldots \dot{\delta} = \frac{\dot{\delta}}{\delta} \right) \ldots \]
Thus, we get for v-g relation:

\[
\vec{v} = \frac{\vec{g}}{4 \pi G \rho_{b} \delta} \frac{1}{\delta \frac{d\delta}{dt}}
\]

Re-write the derivative of density contrast:

\[
\frac{1}{\delta} \frac{d\delta}{dt} = \frac{a}{\delta} \frac{d\delta}{da} \frac{da}{dt} = f(\Omega) H
\]

\[f(\Omega) \simeq \Omega^{0.6}, \quad \Omega = \frac{8 \pi G \rho_{b}}{3 H^{2}}\]

Finally we write:

\[
\vec{v} = \frac{2}{3} \frac{f(\Omega)}{H \Omega} \vec{g}
\]
General equation for growth of perturbations is:

\[ \ddot{\delta} + \frac{4}{3} \frac{\dot{a}}{a} \dot{\delta} = 4\pi G \rho \delta + \frac{\nabla^2 P}{\rho a^2} \]

Particular cases. The simplest case is the flat Universe, wavelength shorter than the horizon, waves longer than the Jeans mass. This is also the case of the cold dark matter (negligible random velocities):

\[ \rho = 0, \quad \Omega_m = 1, \quad \rho_b = \frac{1}{6\pi G a^2} \alpha \frac{a^3}{a^2} \]

Friedmann equation is:

\[ \left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \rho = \left( \frac{2}{3} \right)^2 \]

Now the equation for the density contrast can be written as:

\[ \ddot{\delta} + \frac{4}{3} \frac{\dot{a}}{a} \dot{\delta} = \frac{2}{3} \left( \frac{\dot{a}}{a} \right)^2 \]

Solution of this equation is found in the form \( \delta = A(t)^n \)

This gives:

\[ n(n-1) + \frac{4}{3} n = \frac{2}{3} \Rightarrow n = \frac{2}{3} \text{ or } n = -1 \]

Thus, the general solution is:

\[ \delta(x, t) = A(x) t^{2/3} + B(x) t^{-1} \]

Because in this case \( a \propto t^{2/3} \), we can re-write the growing mode in more elegant form:

\[ \delta_{\text{grow}}(x, t) = \delta_{\text{init}}(x) \frac{a}{a_{\text{init}}} \Rightarrow \delta \propto a \]

Velocity for this mode is found from the continuity equation:

\[ \nabla \vec{v} = -a \dot{\delta} \]

Assuming the form for the density perturbation is \( \delta = A(x) t^{2/3} \), how do we get velocity:

\[ \vec{v} = c(t) \hat{\vec{x}}(x) \quad ? \]

\[ \nabla \vec{v} = -a \frac{\dot{\delta}}{a^2} \Rightarrow c(t) \nabla \hat{\vec{x}} = -a(t) A(x) \frac{2}{3} t^{-1/3} \]

From this we can easily find the time dependence:

\[ c(t) \propto t^{-1} \propto t^{-1/3} \]

LinearGrowth Page 7
Now, normalize the velocity:
Note that $v_{\text{init}}$ and $\delta_{\text{init}}$ are not independent.
We will find their relation later using spectral formalism.

Another case: Open Universe, only non-relativistic matter

$$ P = 0 $$

$$ H^2 = \frac{8\pi G \rho_b}{3} - \frac{\kappa}{a^2}, \quad \text{here} \quad \kappa = -H_0^2(1-\Omega_0) $$

$$ \rho_b = \frac{3H_0^2}{8\pi G} \frac{\Omega_0}{a^3} \rightarrow \Omega(t) = \frac{H_0^2}{H^2} \frac{\Omega_0}{a^3} $$

Introduce new variable:

$$ x \equiv \frac{1-\Omega(t)}{\Omega(t)} = \frac{1-\Omega_0}{\Omega_0} a(t) $$

Now change the time variable in the equation for density contrast:

$$ \frac{d^2 \delta}{dx^2} + \frac{3+4x}{2x(1+x)} \frac{d\delta}{dx} - \frac{3\delta}{2x^2(1+x)} = 0 $$

The growing solution of this simple equation is just wonderful:

$$ \delta = 1 + \frac{3}{x} + \frac{3(1-x)^{1/2}}{x^{3/2}} \ln \left[ \frac{(1+x)^{1/2}}{x^{1/2}} \right] $$

Asymptotic behavior of this solution

$$ x \ll 1 \rightarrow \delta \propto \xi^{2/3} $$

$$ x \gg 1 \rightarrow \delta \sim 1 $$

Fluctuations grow until $x_c \simeq 1$.
Which corresponds to $\alpha \approx \Omega_0$.

Then they stop growing.
Case: **flat Universe with cosmological constant**

Solution for the growth of small perturbations is

\[ \frac{\dot{a}}{a} = \frac{8\pi G}{3} \rho + \frac{\Lambda c^2}{3} \]

\[ \Omega_{\Lambda} = \frac{\Omega_{\Lambda,0}}{H_0^2} \]

\[ x = x_0 a = \left( \frac{\Omega_{\Lambda,0}}{\Omega_0} \right)^{1/3} \]

\[ x_0 = \left( \frac{\Omega_{\Lambda,0}}{\Omega_0} \right)^{1/3} \]

Three models:
1) Flat, matter only
2) Flat + Cosmological constant
3) Open, no cosmological constant

**Left panel:** the same amplitude of fluctuations at early times

**Right panel:** The same amplitude at \( z=0 \)
Case: **waves inside the horizon, relativistic particles dominate**

Growth of perturbations in non-relativistic matter. Fluctuations in the relativistic matter are wiped out by the free streaming.

\[ H^2 = \frac{8\pi G}{3} \left( \rho_m + \rho_\delta \right) \]

\[ \dot{\delta} + 2 \frac{\dot{a}}{a} \delta = 4\pi G \rho_m \delta \]

\[ \rho_m \propto a^{-3}, \quad \rho_\delta \propto a^{-4} \]

\[ \delta = \frac{\rho_m - \langle \rho_m \rangle}{\langle \rho_m \rangle} \]

Note that delta is the density contrast in matter, not in the total density.

Introduce new variable:

Change variable \( t \to y \)

The equation for the growth rate takes the form:

\[ \frac{d^2 \delta}{dy^2} + \frac{2 + 3y}{2y(1+y)} \frac{d\delta}{dy} - \frac{3\delta}{2y(1+y)} = 0 \]

The growing solution of this equation can be found by trying:

\[ \delta = 0 \]

This gives:

\[ \delta_{\text{grow}} = 1 + \frac{3}{2} \frac{y}{a} = 1 + \frac{3}{2} \frac{a}{a_{eq}} \]

Note before the equality the fluctuations grow very little.

After the equality \( \delta \propto a \)
We deal with ideal fluid with pressure $P$. For non-expanding medium the Jeans length was found by equating the time needed for a wave to travel across an object to object's free fall time. Better way: find critical regime for dispersion relation for homogeneous gas with given density and temperature:

For ideal fluid velocity of the sound is given by:

$$v_s = \sqrt{\frac{P}{\rho}} = \sqrt{\frac{\frac{\partial P}{\partial \rho}}{\rho}}$$

For gas before the recombination the pressure is by far provided by radiation

$$P = P_{\text{gas}} + P_\gamma \approx P_\gamma = \frac{n c^2}{3}$$

When pressure waves moves through the gas it creates variations in density and temperature of the radiation:

$$\Delta P_{\text{gas}} \approx \Delta P_\gamma$$

As the result, sound velocity is

$$v_\gamma = \frac{c}{\sqrt{3}}$$

After recombination, photon move freely and gas pressure drops to normal $P = n k T$. This gives sound speed of few km/s.

Growth of fluctuations in gas: