Kinematics of particles in an expanding Universe

Expansion parameter: for freely moving particles and waves all scales change with time proportionally to some universal scaling factor, which we call expansion parameter. It is convenient to normalize the expansion parameter in such a way that:

\[ a = 1 \text{ at present and } a = 0 \text{ at } t = 0 \]

If \( a \) is normalized in this way, then

\[ a = \frac{1}{1 + z} \]  

Is the redshift. For example, if \( l_0 \) is a distance between two galaxies at present, then at moment \( t \) the distance between then is

\[ l(t) = l_0 a(t) \]

Note, that this is valid only for objects, which participate in the global expansion and are not gravitationally bound (or interact) to each other.

Comoving and Proper coordinates: We introduce proper distances as physical distances between objects at some particular time \( t \) or expansion parameter \( a(t) \). Comoving distances are coordinate distances. If at some expansion parameter \( a \) the proper distance was \( l \), then we can extrapolate it to the present moment assuming that the object expands as the whole Universe: \( x = \frac{l}{a} \).

Relation between proper coordinates \( \vec{r} \) and comoving coordinates \( \vec{x} \) are:

\[ \vec{r} = a(t) \vec{x} \quad (\star) \]

In general, comoving coordinates may change with time. This happens when perturbations are present and the Universe is not exactly homogeneous.

Differentiate eq(\( \star \)) with time:

\[ \dot{\vec{r}} = \dot{a} \vec{x} + a \dot{\vec{x}} = \frac{\dot{a}}{a} \left( a \vec{x} \right) + a \dot{\vec{x}} = H \vec{r} + \vec{v}_{\text{pec}} \]

\[ H = \frac{\dot{a}}{a}, \quad \vec{v}_{\text{pec}} = a \dot{\vec{x}} \]

Here \( H \) is the Hubble constant and \( v_{\text{pec}} \) is the peculiar velocity - deviation from the perfect Hubble flow.

Light: Consider two observers at proper separation \( \delta l = a(t) \delta x \). The difference in velocity between the observers is:

\[ \delta v = \frac{\dot{a}}{a} \delta l = H \delta l \]

The first observer sees that the second observer moves away from him. Thus, the Doppler shift of light at the position of the second observer is:
\[ \frac{d\nu}{\nu} = -\frac{\mathbf{S} \cdot \mathbf{v}}{c^2} = -\frac{c}{c} \frac{\mathbf{S} \cdot \mathbf{v}}{c} \]

Light will reach the second observer in time \( \Delta t = \frac{\Delta l}{c} \)

Thus,

\[ \frac{d\nu}{\nu} = \frac{\mathbf{S} \cdot \mathbf{v}}{\Delta t} = \frac{\Delta l}{c} = -\frac{\mathbf{S} \cdot \mathbf{v}}{\Delta t} \Rightarrow \nu = \nu_0 \frac{a_0}{a} \]

We introduce the redshift \( Z \) as:

\[ \frac{\nu_{em}}{\nu_{obs}} = 1 + Z = \frac{1}{a} \]

Here we explicitly used condition \( a_0 = 1 \)

Now, we make the same derivation, but for more general case: motion of any free particle. The particle has a peculiar velocity \( \mathbf{v} \) at the position of the first observer and peculiar velocity \( \mathbf{v}' \) at the position of the second observer. The second observer moves with relative velocity

\[ \mathbf{v}' = \mathbf{v} - \frac{\mathbf{S} \cdot \mathbf{v}}{\mathbf{c}^2} = \mathbf{v} - \frac{\mathbf{S} \cdot \mathbf{v}}{\mathbf{c}^2} \]

The relative velocity of the particle at the second observer is:

\[ \mathbf{v}' = \frac{\mathbf{v} - \mathbf{S} \mathbf{v}}{1 - \frac{\mathbf{S} \mathbf{v}}{\mathbf{c}^2}} \approx \mathbf{v} - \frac{\mathbf{S} \mathbf{v}}{1 - \frac{\mathbf{S} \mathbf{v}}{\mathbf{c}^2}} \]

The change in the peculiar velocity is

\[ \Delta \mathbf{v} = \mathbf{v}' - \mathbf{v} = \frac{\mathbf{S} \mathbf{v}}{1 - \frac{\mathbf{S} \mathbf{v}}{\mathbf{c}^2}} \]

We can write this as:

\[ \frac{d\mathbf{v}}{\mathbf{v}} = \frac{\mathbf{S} \mathbf{v}}{\mathbf{v}(1 - \frac{\mathbf{v}^2}{\mathbf{c}^2})} = -\frac{\mathbf{S} \mathbf{v}}{\mathbf{a}} \]

Introduce momentum of a particle:

\[ P \equiv \frac{\mathbf{v}}{\sqrt{1 - \frac{\mathbf{v}^2}{\mathbf{c}^2}}} \]

The eq(\( \star \)) is a full derivative: \( dp/p \). Thus, the equation can be integrated over time:

\[ P = P_0 / a \]
For non-relativistic particles: \( \nu \ll c \Rightarrow p = \nu \quad \nu_{\text{pec}} \propto \frac{1}{a} \)

Conclusion: peculiar velocities, which are not supported by perturbations in gravity, decay as the Universe evolves.

For relativistic particles:
\[
E = cp = h\nu \propto \frac{1}{a}
\]

Changes in the temperature of radiation: use the first law of thermodynamics:
\[
dE = TdS - pdV, \quad dS = 0
\]
\[
E = \int c^2 V, \quad p = \frac{6T^4}{c^2},
\]
\[
p = \frac{\rho c^2}{3}
\]

Thus,
\[
\frac{dp}{\rho} = -\frac{4}{3} \frac{dV}{V} = -\frac{4da}{a}
\]
\[
\Rightarrow \rho \propto a^{-4} \quad T = \frac{T_0}{a} = T_0 (1+z)
\]

Conclusions:
- The contribution of relativistic particles (e.g. photons) declines with the expansion parameter faster than the contribution of non-relativistic particles. Thus, in the past relativistic particles such as photons and neutrinos played much more important role than they do today.
- The temperature of radiation at given redshift does *not* depend on how the Universe was expanding and what it is made of. Lambda or no lambda, curvature or no curvature - this all does not affect the temperature. (It does affect the mapping from the redshift to real time, though). An important assumption is that the number of photons in a comoving volume is preserved. This may not be true at every stage of the evolution of the Universe.