**Corrections due to General Relativity and relativistic effects**

Logic: Use FRW metric, assume a simple form of energy-stress tensor, get Friedmann equation.

FRW: \[ ds^2 = c^2 dt^2 - a^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right] \]

Einstein's equations:
\[ R_{ij} - \frac{1}{2} g_{ij} R = \frac{8\pi G}{c^2} T_{ij} \]

Ricci tensor:
\[ R_{ij} = -3 \left( \frac{\dot{a}}{a} \right)^2 \]

Scalar curvature
\[ R = -6 \left( \frac{\ddot{a}}{a} + \left( \frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} \right) \]

Energy-stress tensor:
\[ T_{ij} = \left( \rho c^2 + P \right) u_i u_j - g_{ij} \rho + \mathbf{A} g_{ij} \]

\[ \mathbf{u} = \frac{d\mathbf{a}}{ds} = \begin{cases} \frac{1}{\sqrt{1 - v^2/c^2}} \\ \frac{v}{\sqrt{1 - v^2/c^2}} \end{cases} \]

The 0-0 component of Einstein's equations gives:
\[ \left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \rho - \frac{K}{a^2} \]

The ii components give:
\[ 2 \frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a} = -8\pi G \rho - \frac{K}{a^2} \]

Combining those equations, we get:
\[ \frac{\ddot{a}}{a} = -\frac{4\pi G}{3} \left( \rho + \frac{3P}{c^2} \right) \]

The 0-0 equation also can be written as:
\[ \rho \dot{a}^3 = \frac{3}{8\pi G} a (\dot{a}^2 + k) \]

Differentiate it relative to time and use ii:
\[ \frac{d}{dt} \left( \rho a^3 \right) = \frac{3}{8\pi G} \left[ \dot{a} (\dot{a}^2 + k) + 2a \dot{a} \ddot{a} \right] = \frac{3}{8\pi G} a^2 \dot{a} \dot{a} \frac{8\pi G \rho = -\rho \frac{d a^3}{d t}}{\dot{a}^3} \]

This is the first law of thermodynamics: \[ dE = -\rho dV \]

We can re-write it in the form of continuity equation
\[ \frac{d\rho}{\dot{a}^3} + 3H (\rho + \frac{P}{c^2}) = 0 \]
For non-relativistic particles $P$ is much smaller than $\rho c^2$. Thus, it can be neglected. This gives usual $\rho \propto a^{-3}$. For relativistic particles such as photons or neutrinos $P = \frac{\rho c^2}{3}$.

There are more exotic particles and fields, which can give different relations between the pressure and density. Conventionally the equation of state is written in the form:

$$P = w \rho c^2$$

Parameter $w$ is a very important for the evolution of the Universe:

- $w = 0$ for non-relativistic particles
- $w = \frac{1}{3}$ Relativistic particles
- $w = 1$ mass-less scalar fields
- $w = -\frac{1}{3}$ Curvature. Cosmic strings (vacuum energy in 1d defects)
- $w = -\frac{2}{3}$ Domain walls (vacuum energy trapped in 2d defects)
- $w = -1$ Cosmological constant. Massive scalar fields

**Case $k=0$,** $P = \frac{\rho c^2}{3}$ Radiation dominated epoch.

In this case:

$$\frac{\dot{a}}{a} = -\frac{4\pi G}{3} \left( \rho + \frac{3P}{c^2} \right) = -\frac{8\pi G \rho}{3}$$

From continuity equation we get:

$$\rho = \rho_0 a^{-4}$$

Substitute this into the Friedmann equation and integrate it:

$$a = \left( \frac{32 \pi G \rho_0}{3} \right)^{\frac{1}{4}} t^\frac{1}{2} \Rightarrow \rho = \frac{3}{32 \pi G t^2}$$