Potentials for Non-spherical Distributions
Non-spherical systems: examples, which demonstrate general trends

In general, non-spherical systems are much more difficult to handle. Still, we can solve some simple and astronomically interesting cases.

1. Thin infinite disk: a simple model for stellar disks

   \[ \sigma \text{ is the surface density} \]
   \[ \Delta S = \text{area} \]
   \[ \hat{n} = \text{unit vector orthogonal to the surface} \]

   Because of the plane symmetry, the only non-zero component of \( \vec{\Phi} \) is \( g_2 \).

   Use the Gauss law to find \( g_2 \):

   \[
   \oint \vec{\Phi} \cdot d\vec{S} = -4\pi G M \implies M = \sigma \Delta S \]

   \[
   \oint \vec{\Phi} \cdot d\vec{S} = -2\pi G \sigma \Delta S \]

   Now, from \( \frac{\partial U}{\partial z} = -g_2 \) find \( U = 2\pi G \sigma G^2 / 2 \).

   If we start with the Poisson equation, we get the same answer. In this case \( \rho = \sigma S(2) \).

Note that in this case we cannot normalize \( U \) in usual way to have \( U(\infty) = 0 \). Instead, we use \( U(0) = 0 \).

Another effect: in spite of infinite density at \( z=0 \), both \( g_2 \) and \( U \) are finite.
Slightly more complicated system: **Thick disk**

\( \rho_0 = \text{const} = \text{density of the disk} \)

\( 2L = \text{disk height} \)

\( \Sigma = 2 \rho_0 L = \text{surface density} \)

\[ M(\ell) = \text{mass inside the cylinder within } |\ell| \]

\[ M(\ell) = \begin{cases} 
\Sigma 4 \ell S, & \text{for } |\ell| > L \\
2 \rho_0 |\ell| 4 \ell S, & \text{for } |\ell| \leq L 
\end{cases} \]

\[ \int_0^\ell \rho \, ds = -4 \pi G \rho_L \Rightarrow g(\ell) = \begin{cases} 
-4 \pi G \rho_L \ell, & \ell > L \\
-4 \pi G \rho_L 2, & 0 < \ell < L 
\end{cases} \]

\[ U = \begin{cases} 
\frac{2}{3} G \rho_0 L^2, & \ell > L \\
\frac{2}{3} G \rho_0 \ell^2 + U(0), & \ell \leq L 
\end{cases} \]

The constant should be chosen in such a way, that \( U \) is a continuous function \( \Rightarrow U(0) = \frac{2}{3} G \rho_0 L^2 \)

\[ U(L) = 2 U(0) \]
Filament with constant density

The filament is infinite; radius is R; density is \( \rho_0 \)

\( S(r) = \frac{\pi \rho_0}{2} \left\{ \begin{array}{ll}
\pi r^2, & \text{for } r < R \\
\pi R^2, & \text{for } r \geq R
\end{array} \right. \)

Apply the Gauss law: \( \oint \mathbf{E} \cdot d\mathbf{S} = -4\pi G M \)

\( g_r \, dl 
\times r = -4\pi G S(r) \, dl \)

This gives:

\[ g_r = -\frac{2 \pi G S(r)}{r} \left\{ \begin{array}{ll}
-\frac{4\pi G \rho_0 R}{r}, & \text{for } r < R \\
-\frac{4\pi G \rho_0 R^2}{r}, & \text{for } r \geq R
\end{array} \right. \]

\[ U(r) = \left\{ \begin{array}{ll}
\pi G \rho_0 R^2 + U(0), & \text{for } r < R \\
\pi G \rho_0 R^2 + 2\pi G \rho_0 R^2 \ln\left(\frac{r}{R}\right) + U(0), & \text{for } r \geq R
\end{array} \right. \]

We can find the potential for a string with finite mass by setting \( R \to 0 \) and keeping \( G = \frac{\pi \rho_0 R^2}{\text{const}} = G_0 \)

In this case

\[ U = 8G_0 \rho_0 R \ln r \]

This goes to \( \infty \) both at \( r \to 0 \) and at \( r \to \infty \)
Homogeneous Ellipsoid: $\rho = \rho_0$ is constant inside ellipsoid:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Gravitational potential inside the ellipsoid is given by this expression.

Here coefficient $A$, $B$, and $C$ are:

$$A = abc \int_0^{\infty} \frac{dz}{(a^2 + z)^{1/2}}$$
$$B = abc \int_0^{\infty} \frac{dz}{(b^2 + z)^{1/2}}$$
$$C = abc \int_0^{\infty} \frac{dz}{(c^2 + z)^{1/2}}$$

**Analysis of these relations**

Use the Poisson equation

This gives the relation between $A$, $B$, $C$

$$\nabla^2 U = 4\pi G \rho (x, y, z)$$

$$A + B + C = \lambda$$

For an oblate ellipsoid

the coefficients $A$, $B$, and $C$ are

$$A = B = \frac{(1-e^2)^{1/2}}{e^2} \left[ \frac{\arcsin e - (1-e^2)^{1/2}}{e} \right]$$

$$C = 2 \left( \frac{(1-e^2)^{1/2}}{e^2} \right) \left[ \frac{1}{(1-e^2)^{1/2}} - \frac{\arcsin e}{e} \right]$$

where $e = 1 - \frac{c^2}{a^2}$

at small $e \ll 1$ \quad $\arcsin e \approx e + \frac{e^3}{3!}$ \quad $A = B = \frac{2}{3}$

at $e = 1$ \quad $\arcsin (1) = \frac{\pi}{2}$ \quad $A \approx \frac{\pi}{2} - \sqrt{1-e^2} = \frac{\pi}{2} - \frac{c}{a} \rightarrow 0$

$C \approx 2$

so, in this limit \quad $(e=1, a \gg c)$ \quad $U \approx 2\pi G \rho_o \frac{c^2}{a}$
How much the acceleration changes if we flatten the distribution?

\[ \frac{\partial U}{\partial x} \bigg|_{x=a, y=0, z=0} = \pi G \rho_0 A x \]

Rewrite it in a different way:

Mass is equal to

\[ M = \frac{4\pi}{3} abc \rho_0 \]

Compare acceleration of a sphere of the same mass with the real acceleration

\[ g_{x, \text{sphere}} = \frac{GM}{a^2} \iff g_x = \pi G \rho_0 A x \]

For \[ a = 1 \implies A = \frac{\pi}{2} \sqrt{1-e^2} = \frac{\pi}{2} \cdot \frac{c}{a} \implies g_x = \pi G \rho_0 c \]

Thus, the largest error is

\[ \frac{g_{x, \text{sphere}}}{g_x} \leq \frac{1}{2.356} \]

The acceleration at point

\[ x = y = 0, z = c \]

This is almost the same acceleration as at point

\[ x = a, z = 0 \]
Spheroidal with inhomogeneous density distribution

logic: start with a thin shell (homoeoid) =>
Find the potential => integrate over all
shells => get potential
This is the same logic we had for spherical
systems, but now the shell is not a spherical

**Homoeoid**: constant density between two
similar spheroids:

\[
\frac{R^2}{a^2} + \frac{Z^2}{b^2} = 1 \quad \text{and} \quad (x^2 + y^2)^{1/2} = (R^2 + Z^2)^{1/2}
\]

\[
\Rightarrow \text{Exterior isopotential surfaces of a homoeoid are spheroids, that are conjugate with the shell.}
\]

\[
\Rightarrow \text{Inside the shell the potential is constant}
\]

**Spheroids with iso-density surfaces**: (similar spheroids)

\[
m^2 = R^2 + \frac{Z^2}{1-e^2}
\]

\[
\Rightarrow V = V(m, e)
\]

Volume \(V = \frac{4}{3} \pi a^2 b\)

Interesting application:

\[
\rho(m^2) = \rho_0 \left[ 1 + \left( \frac{m^2}{a_0^2} \right)^{3/2} \right]
\]

\(a_0 = \text{core radius} \quad \frac{dV}{dr} = \frac{dV}{dr} \]

\(e = \text{eccentricity} \quad \frac{dV}{dr} \]

This gives

\[
V_c(R) = \frac{4}{3} \pi \rho_0 a_0^3 \frac{1-e^2}{R} \int \frac{F(a, k)-E(a, k)}{a^2} \quad \text{icomplete elliptical integrals}
\]

\[
k = \left[ \frac{(ae)^2}{1} + 1 \right]^{-1/2}
\]
Another example: exponential thin disk

Again, the final expression is not easy.

For a thin disk with surface density $\Sigma(R)=\Sigma_0 e^{-R/R_0}$

$$v_e^2(R) = 4\pi G_0 \mu \cdot g^2 \left[I_0(y) K_0(y) - I_1(y) K_1(y)\right]$$

\[ y = \frac{R}{R_{Re}} \]

$\mu$, $K_i$, $K_0$ - Bessel functions
Figure 2-17. The circular-speed curves of: an exponential disk (full curve); a point with the same total mass (dotted curve); the spherical body for which $M(r)$ is given by equation (2-170) (dashed curve).
Figure 2-13. The ellipticity $\epsilon_\Phi$ of an equipotential surface versus the surface's semi-major axis length $r$. Each curve is labeled by the ellipticity $\epsilon_\rho = 1 - q$ of the body with density (2-92) that generates the corresponding potential. Notice the rapidity with which the equipotential surfaces become spherical at large $r/a_0$. 
Figure 2-12. Circular speed versus radius for three bodies with the same face-on projected density profile (the modified Hubble profile) but different axis ratios $q = b/a$. Though all three bodies have the same total mass inside a spheroid of given semi-major axis, $v_c$ increases with flattening $1 - q$. 