Equations of hydrodynamics: eulerian and lagrangian approaches
Hydrodynamics

What are the conditions for using equations of hydrodynamics?

- Scales are much larger than the mean-free-path of particles in the medium.
- Time-scales are longer than the time-scale of relaxation.
- Volumes are larger than some 'elementary volume': volume is large enough to have many particles inside it and, at the same time, small enough to have small gradients across the 'elementary volume'.

Thermal velocities of particles: \( u_{\text{thermal}} = \left( \frac{3kT}{m} \right)^{1/2} \)

For a neutral gas the mean-free-path \( \ell = \frac{1}{s} \) where \( s \) is the cross-section of atoms.

For air: \( s = 4.4 \times 10^{-6} \text{ cm}^2 \) \( n = 10^{19} \text{ cm}^{-3} \)

For plasma: two-body relaxation time for electron-proton collisions is

\[ t_{\text{ep}} = \frac{3}{4} \left( \frac{m_e}{m_p} \right)^{1/2} \left( \frac{kT}{2\pi} \right)^{3/2} \frac{1}{e^4 n_e \ln \Lambda} \approx 0.275 \frac{T^{3/2}}{n_e \ln \Lambda} \]

Here \( m_e = \frac{m_p}{1836} \) mass of an electron, \( e = 1.6 \times 10^{-19} \text{ electron charge} \)

\( n_e = \text{number density of electrons} \)

\( \ln \Lambda = \text{Coulomb logarithm (\approx 0-20)} \)
<table>
<thead>
<tr>
<th>System, room Temp.</th>
<th>Parameters</th>
<th>75th</th>
<th>mean-free path</th>
</tr>
</thead>
<tbody>
<tr>
<td>Air</td>
<td>$n_e = 10^{19} \text{ cm}^{-3}$, $T = 300 \text{ K}$, $\sigma = 4.4 \times 10^{-16} \text{ cm}^2$, $m = 29 \text{ pm}$</td>
<td>$5 \times 10^4 \text{ cm/s} = 0.5 \text{ km/s}$</td>
<td>$2 \times 10^{-4} \text{ cm}$</td>
</tr>
<tr>
<td>ISM, disk, galaxy</td>
<td>$n_e = 1 \text{ cm}^{-3}$, $m_e$, $T = 10^4 \text{ K}$</td>
<td>$7 \times 10^7 \text{ cm/s} = 700 \text{ km/s}$</td>
<td>$10^{12} \text{ cm} = 3 \times 10^7 \text{ pc}$, $t_e - p = 10^4 \text{ sec}$</td>
</tr>
<tr>
<td>Clusters, gas</td>
<td>$n_e = 10^{-3}$, $T = 10^8 \text{ K}$, electrons</td>
<td>$7 \times 10^9 \text{ cm} = 0.25 \text{ c}$</td>
<td>$10 \text{ cm} = 30 \text{ kpc}$</td>
</tr>
</tbody>
</table>

$2^3$ $t_e - p = 1.4 \times 10^5 \text{ yrs}$

$= 4 \times 10^5 \text{ yrs}$
Equations of hydrodynamics:

Some math. relations

\( \vec{a} \) is a vector field; \( b \) is a scalar field.

If volume \( V \) is surrounded by closed surface \( S \), then

\[
\oint_S \vec{a} \cdot d\vec{s} = \iiint_V \text{div} \, \vec{a} \cdot dV
\]

\[
\oint_S b \, d\vec{s} = \iiint_V \text{grad} \, b \cdot dV
\]

\[
\frac{d}{dt} = \frac{\partial}{\partial t} + (\vec{V} \cdot \nabla)
\]

Take small cube.

\[
\frac{\partial}{\partial x} (x+dx, y, z) \approx \vec{a}(x, y, z) + \frac{\partial \vec{a}}{\partial x} \Delta x
\]

\[
\sum_{\text{all surfaces}} \sum_{\text{cube}} (a_x \Delta S_x + a_y \Delta S_y + a_z \Delta S_z)
\]

\[
= a_x (x+dx, y, z) \Delta S_x - a_x (x, y, z) \Delta S_x + a_y (x, y+dy, z) \Delta S_y - a_y (x, y, z) \Delta S_y + a_z (x, y, z+dz) \Delta S_z - a_z (x, y, z) \Delta S_z
\]

\[
= \frac{\partial a_x}{\partial x} \Delta x \Delta S_x + \frac{\partial a_y}{\partial y} \Delta y \Delta S_y + \frac{\partial a_z}{\partial z} \Delta z \Delta S_z = \text{div} \, \vec{a} \, dV
\]

Approximate volume \( V \) by many small boxes with \( dV \) each, count contributions of small boxes and set \( dV \to 0 \)

\[
\sum b \, d\vec{s} = \oint_S b \, d\vec{s} = \frac{\partial}{\partial x} a_x \Delta S_x + \frac{\partial}{\partial y} a_y \Delta S_y + \frac{\partial}{\partial z} a_z \Delta S_z
\]

\[
= \frac{\partial}{\partial x} \text{grad} \, b \cdot dV
\]

Fluid element changes its position by \( \frac{dx}{dt} \) in \( dt \), any property \( f \) of the element also changes.

\[
\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial}{\partial x} \frac{df}{dx} + \frac{\partial}{\partial y} \frac{df}{dy} + \frac{\partial}{\partial z} \frac{df}{dz} \Rightarrow \frac{df}{dt} = \frac{\partial f}{\partial t} + (\vec{V} \cdot \nabla) f
\]
Equations of hydrodynamics: - Euler approach
- Lagrange approach

**Euler:** Find evolution of parameters of fluid at each point in space and time

![Euler diagram](image)

**Lagrange:** Find evolution of each fluid element:

![Lagrange diagram](image)

**Euler formalism:** equations of hydrodynamics

**Continuity equation.** Take volume element \( V \). Mass inside \( V \) is

\[
\int p \, dv
\]

Through a surface element \( dS \) of the volume flows out mass \( \rho \overrightarrow{v} \cdot dS \). Total mass leaving the volume \( V \) in \( dt \) is

\[
\int \rho \overrightarrow{v} \cdot dS \, dt
\]

The total rate of change of mass is

\[
\frac{\partial}{\partial t} \left[ \int p \, dv \right] = -\int \rho \overrightarrow{v} \cdot dS \quad \text{Note the sign}
\]

Change the integral on the right from surface to volume and change the order of integration and time derivative

\[
\int \left( \frac{\partial p}{\partial t} + \nabla (\rho \overrightarrow{v}) \right) \, dV = 0
\]
This relation must be valid for any volume \( V \). This can be true only if the integrand is equal to zero.

\[
\frac{\partial \rho}{\partial t} + \nabla (\rho \vec{v}) = 0
\]

Rewrite this equation:

\[
\frac{\partial \rho}{\partial t} + \nabla \rho + \rho \nabla \vec{v} = 0
\]

The first two terms are equal to \( \frac{\partial \rho}{\partial t} + \nabla \rho = \frac{\partial \rho}{\partial t} \)

Thus,

\[
\frac{\partial \rho}{\partial t} + \rho \nabla \vec{v} = 0
\]

For incompressible fluid (e.g. water) \( \frac{\partial \rho}{\partial t} = 0 \). In this case \( \text{div} \vec{v} = 0 \).

This tells us that \( \text{div} \vec{v} \) is a measure of compressibility of fluid.

**Euler equation.** Take volume \( V \). Pressure force acting on the volume from fluid, which surrounds the volume is

\[
(*) \quad \int_{\partial V} -\vec{F} \cdot d\vec{s}
\]

Note the sign

\[
\int_{\partial V} \vec{F} \cdot d\vec{s} = \int_{\partial V} \text{grad} \rho \cdot d\vec{v}.
\]

The rate of change of the momentum of fluid inside \( V \) is

\[
(**) \quad \int_{V} \frac{\partial \vec{v}}{\partial t} \cdot d\vec{v}.
\]

This gives \( \frac{\partial \vec{v}}{\partial t} = -\text{grad} \rho \)

Change \( dt \) to \( d\tau \):

\[
\frac{\partial \vec{v}}{\partial \tau} + (\vec{v} \cdot \nabla) \vec{v} = -\int_{\partial V} \text{grad} \rho - \nabla \Phi
\]

where \( \Phi \) is the grav. potential.
Energy equation

We start with the first law of thermodynamics:

\[
\frac{dE}{dt} + p \frac{dV}{dt} = \frac{dQ}{dt}
\]

Here, \( E \) is internal energy per unit mass, \( V = \frac{1}{\rho} = \) specific volume, \( dQ \) is external energy per unit mass. Using the continuity and Euler equations, the energy equation can be written in the form

\[(\star) \quad \frac{\partial}{\partial t} \left( \rho E + \frac{\rho V^2}{2} \right) + \nabla \left( \rho \left( E + \frac{V^2}{2} \right) + P V \right) = \rho \frac{dQ}{dt}\]

We note that \( (E + \frac{V^2}{2}) \) is the total energy = thermal + kinetic

\[ E = \rho \left( \frac{V^2}{2} + E \right) = \text{total energy per unit volume} \]

\[(\star\star) \quad \frac{\partial E}{\partial t} + \nabla \cdot (E \mathbf{V}) = - \nabla \cdot (P \mathbf{V}) + \rho \frac{dQ}{dt}\]

For an adiabatic process, we can use the equation of entropy conservation: \( \frac{dS}{dt} = 0 \),

where \( S \) is the specific entropy. This can be written in the form

\[ \nabla S + \mathbf{V} \cdot \nabla S = 0 \]
Lagrangian approach to hydrodynamics

Follow motion of a fluid element, mass of the element is preserved, but its position in space and its volume change as the element moves.

We must introduce a way to identify the element and to distinguish it from other elements. Let $\tilde{q}$ be a label ("coordinate") which characterizes the element. There are different ways of choosing $\tilde{q}$. For example, $\tilde{q}$ can be position of the element at some initial moment of time.

Motion of the fluid is described by the following set of relations:

\begin{align*}
\rho &= \rho(t, \tilde{q}) = \text{density} \\
\mathbf{v} &= \mathbf{v}(t, \tilde{q}) = \text{velocity} \\
\mathbf{p} &= \mathbf{p}(t, \tilde{q}) = \text{pressure} \\
\mathbf{x} &= \mathbf{x}(t, \tilde{q}) = \text{position in space}
\end{align*}

\[ \text{samples: (i) Plane motion: there is only one coordinate} \ x \]

Mass $m$ between (any) elements is preserved, but position of the elements changes with time. We can use $m$ as Lagrangian coordinate $\tilde{q}$. 
For simplicity we assume that the first element does not move.

\[ m(x) = \int_{x_1}^{x} \rho \, dx, \quad dm = \rho \, dx \]

Mass \( m \) is our independent variable.

The continuity equation in Eulerian coordinates is

\[ \frac{\partial \rho}{\partial t} + \text{div}(\rho \vec{u}) = 0 \]

We need to rewrite it so that time derivative is \( \frac{d}{dt} \), not \( \frac{\partial}{\partial t} \):

\[ \frac{\partial \rho}{\partial t} + \text{div}(\rho \vec{u}) = \frac{d\rho}{dt} + \rho \text{div} \vec{u} = \frac{d\rho}{dt} + \rho \text{div} \vec{u} = 0 \]

In 1D case this equation takes form:

\[ \frac{d\rho}{dt} + \rho \frac{\partial \vec{u}}{\partial x} = 0 \]

Now we need to change variables from \( x \) to \( u \):

use \( dm = \rho \, dx \). We get continuity equation in Lagrangian coordinates:

\[ \frac{1}{\rho^2} \frac{d\rho}{dt} = -\frac{\partial \rho}{\partial m} \]

The equation of motion in Lagrangian coordinates is

$$\frac{d\mathbf{v}}{dt} = -\nabla \mathbf{p} - \mathbf{v} \frac{\partial \mathbf{v}}{\partial \mathbf{p}}$$

Here we need to change \( \frac{\partial}{\partial x} \) to \( \frac{\partial}{\partial m} \):

$$\frac{1}{\rho} \frac{\partial \rho}{\partial x} = \frac{\partial f(m, t)}{\partial m} \quad \frac{\partial \rho}{\partial x} = \rho \frac{\partial f(m, t)}{\partial m}$$

Thus, the final form of the equation of motion in Lagrangian coordinates takes the form

$$\frac{d\mathbf{v}}{dt} = -\nabla \mathbf{p} - \rho \frac{\partial f}{\partial m}$$

**Energy equation**

$$\frac{ds}{dt} + \rho \frac{d\mathbf{v}}{dt} = \frac{dQ}{dt}$$

For adiabatic process: \( \frac{ds}{dt} = 0 \)

Another choice of coordinates: use initial coordinate \( x \) as the Lagrangian coordinates. If \( \rho_0(x) \) is the density distribution at initial moment, then mass conservation

$$\rho_0 \, dx = \rho(r, t) \, dr$$

Equations of hydrodynamics are:

$$\frac{1}{\rho^2} \frac{d\rho}{dt} = -\frac{\partial f(t, x)}{\partial x} \frac{1}{\rho_0(x)}$$

$$\frac{d\mathbf{v}(t, x)}{dt} = -\frac{1}{\rho_0(x)} \frac{\partial \mathbf{f}(t, x)}{\partial x} - \frac{\rho(x, t) \partial f}{\rho_0(x) \partial x}$$
Example: motion of cold gas in one-dimensional case

\[ \rho = 0, \quad \nabla = 0 \]

Initial conditions:

\[ V = V_0 \sin(kx), \quad \nabla = 0 \text{ and } k \text{ are constants} \]

\[ \rho = \rho_0 \quad \text{initial density } \rho = \rho_0 = \text{const} \]

How the gas flows at later times?

\[ \frac{1}{\rho^2} \frac{d\rho}{dx} = - \frac{\partial V}{\rho_0 \partial x} = - \frac{V_0 k \cos(kx)}{\rho_0} \]

The r.h.s. does not depend on time \( t \). Thus, we can integrate this equation.

\[ \frac{1}{\rho_0} \frac{1}{\rho(x,t)} = - \frac{V_0 k \cos(kx)}{\rho_0} (t-t_0) \]

\[ \Rightarrow \quad \rho = \frac{\rho_0}{1 + \frac{V_0 k (t-t_0) \cos(kx)}{\rho_0}} \]

Find relation between Lagrangian coordinate \( x \) and Eulerian coordinate \( r \): From continuity equation

\[ \rho(x,t) \, dr = \rho_0(x) \, dx \Rightarrow \frac{dr}{dx} = \frac{\rho_0}{\rho(x,t)} \]

Integrate over \( x \):

\[ r = x + V_0 (t-t_0) \sin(kx) \]
Early moments of time:
\[ V_0 k (t - t_0) \ll 1 \]

Critical moment:
\[ V_0 k (t - t_0) = 1 \]
infinite density at one point
Column density plot of Cloud 1 in the $y$-$z$ plane at $t = 19.1$ Myr, integrating over the central 16 pc along the $x$-direction. The dots show the stellar objects (sink particles). The electronic version of this figure shows an animation of this region from $t = 16.6$ to 19.9 Myr. The column density is in code units, which correspond to $9.85 \times 10^{19}$ cm$^{-2}$.

Model of star formation in Molecular clouds