Bondi accretion
Hydrodynamical equations for a spherical system: Bondi flow

\[ \frac{2p}{dt} + \frac{1}{r^2 \sigma} (r^2 p u_r) = 0 \]

\[ \frac{\partial u_T}{\partial t} + u_r \frac{\partial u_r}{\partial r} - \frac{u_T^2 + u_r^2}{r} = -\frac{1}{r^2 \sigma} \frac{\partial p}{\partial r} - \frac{\partial \Phi}{\partial r} \]

For a stationary flow \( \frac{2p}{dt} = 0 \); assume that \( u_T = u_r = 0 \)

This gives

\[ \frac{1}{r^2 \sigma} (r^3 \rho u_r) = 0 \quad \text{\( \Rightarrow \) \( \Phi = \text{const} \)} \]

\[ 4\pi r^2 \rho \sqrt{\rho} = M = \text{const} \quad \text{acceleration rate} \]

Gravity is defined by the mass of the central object \( M \):

\[ \frac{\partial \Phi}{\partial r} = \frac{GM}{r^2} \]

(2) is rewritten as

\[ \frac{\partial \Phi}{\partial r} = \frac{GM}{r^2} \]

\[ \frac{2p}{\partial r} + \frac{1}{r^2} \frac{\partial p}{\partial r} + \frac{GM}{r^2} = 0 \]

(3)

Polytropic gas: \( p = A \rho^\gamma, \quad A = \text{const} \)

\[ \frac{\partial p}{\partial r} = \frac{2p}{\partial r} \frac{\partial \rho}{\partial r} = \frac{\gamma}{\rho} \frac{\partial p}{\partial r} \quad \Rightarrow \quad \frac{\partial p}{\partial r} = \frac{\gamma}{\rho} \frac{\partial p}{\partial r} \]

Continuity equation gives

\[ r^2 \rho \frac{\partial \rho}{\partial t} + \rho \frac{\partial (r^2 u_r)}{\partial r} = 0 \quad \Rightarrow \quad \frac{\partial p}{\partial r} = -\frac{\partial (r^2 u_r)}{\partial r} \]

Thus, eq (3) can be rewritten in the form

\[ \frac{1}{2} \frac{\partial (r^2 u_r)}{\partial r} - \frac{y^2}{2} \frac{\partial (r^2 u_r)}{\partial r} + \frac{GM}{r^2} = 0 \]

\[ \Rightarrow \frac{1}{2} \frac{\partial (r^2 u_r)}{\partial r} - \frac{v^2}{2} \frac{\partial v}{\partial r} + \frac{GM}{r^2} = 0 \quad \Rightarrow \quad \frac{1}{r} \left( 1 - \frac{v^2}{c^2} \right) \frac{\partial (\frac{2c^2}{v^2} - \frac{v^2}{2})}{\partial r} = -\frac{GM}{r^2} \left( 1 - \frac{2c^2}{v^2} \right) \]
This equation has 6 types of possible solutions, but not all of them are physical and not all of them correspond to accretion.

For accretion, we expect that at very large radii, the gravitational pull from the central mass is negligible. Thus, gas should be at rest at large radii: $V(\infty) = 0$.

Then the velocity increases as we come closer to the central mass: $\frac{du^2}{dr} < 0$.

At large $r$, the term in $[13]$ on the r.h.s. is negative, which means that r.h.s. is positive. Because $\frac{du^2}{dr} < 0$, in order to have a positive r.h.s. the velocity must be $v^2 > \frac{G^2}{2} R_c^2$.

Thus, physical solution for accretion regime is

Subsonic at large radius

But at small radii, $1 - \frac{2G^2}{R_c^2}$ changes sign and becomes positive. This means that $GMv^2 > \frac{G^2}{2} R_c^2$.

Critical radius: $V_c = c_s$ \[ V_c = V_{\text{crit}} = \frac{GM}{2c_s^2 R_{\text{crit}}} \]

At small radii, the flow is supersonic.

Critical point: $V = c_s$ \[ V = V_{\text{crit}} = \frac{GM}{2c_s^2 R_{\text{crit}}} \]

$R_{\text{crit}} = \frac{64}{15} \frac{1}{c_s^2} \approx 7.5 \times 10^4 \left( \frac{10^4}{10^8} \right) \text{ cm} \ll \text{much larger than typical size of accreting system object}$
Possible solutions (not all are for accretion)

1. \( v^2 = \xi^2 \) at \( r = R_{\text{crit}} \)
   \( v^2 \to 0 \) as \( r \to \infty \)
   \( v^2 < \xi^2 \) for \( r > R_{\text{crit}} \)
   \( v^2 > \xi^2 \) if \( r < R_{\text{crit}} \)

(reverse situation to accretion)

2. \( v^2 = \xi^2 \) at \( r = R_{\text{crit}} \)
   \( v^2 \to 0 \) as \( r \to 0 \)
   \( v^2 > \xi^2 \) for \( r > R_{\text{crit}} \)
   \( v^2 < \xi^2 \) at \( r < R_{\text{crit}} \)

(stellar wind, Parker solution)

3. \( \frac{\partial v^2}{\partial r} = 0 \) at \( r = R_{\text{crit}} \)
   \( v^2 < \xi^2 \) for all radii

4. \( \frac{\partial^2 v}{\partial r^2} = 0 \) at \( r = R_{\text{crit}} \)
   \( v^2 > \xi^2 \) for all radii

5. \( \frac{\partial v}{\partial r} = 0 \) at \( r = R_{\text{crit}} \)
   \( v^2 = \xi^2 \)
   \( (v^2 = \xi^2) \) for \( r > R_{\text{crit}} \) always
   \( \text{unphysical:} \) two velocities at the same radius

6. \( \frac{\partial^2 v}{\partial \nu^2} = 0 \) at \( r = R_{\text{crit}} \)
   \( r > R_{\text{crit}} \) always
   \( r < R_{\text{crit}} \) always
Equation 
\[ \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{\phi}{r^2} = 0 \]

can be integrated once:

\[ \frac{\phi^2}{2} + \frac{\phi_0^2}{2} - \frac{GM}{r} = \text{const} \]

The constant of integration can be found using boundary condition at 
\( r \to \infty, \phi = 0 \) \( \phi = \phi_0(\infty) \)

\[ \frac{\phi^2}{2} + \frac{\phi_0^2}{2} - \frac{GM}{r} = \phi_0^2(\infty) \]

\[ \frac{\phi^2}{2} = \frac{\phi_0^2}{2} - \frac{GM}{r} \]

Using this equation (Boussinesq) we can relate parameters at \( r \to \infty \) with parameters at the sonic flow:

\[ \Gamma_{\text{crit}} = \frac{GM}{\phi_0^2(\infty)} \frac{(5-3\delta)}{4} \]

\[ \phi_{\text{crit}} = 0 \]

\[ \text{Note: } \Gamma_{\text{crit}} = 0 \text{ for } \delta = \frac{1}{2} \]

and the sound velocity at the critical (sonic) point is

\[ \phi_{\text{sonic}}^2 = \frac{2}{5-3\delta} \]

The density at the critical point is

\[ \rho_{\text{crit}} = \rho_0 \left( \frac{\phi_{\text{crit}}}{\phi_0} \right)^{\frac{2}{\delta-1}} \]

\[ \rho_{\text{crit}} = \rho_0 \left( \frac{\phi_{\text{sonic}}}{\phi_0} \right)^{\frac{2}{\delta-1}} \]

Accretion rate: \( \dot{M} = 4\pi \rho^2 \dot{\phi} \) can be found:

\[ \dot{M} = 4\pi \rho^2 \dot{\phi} \rightarrow \dot{M} = \frac{G^2 M^2}{\phi_0^2 \phi^2} \left[ \frac{\phi}{\phi_0} \right] \left( \frac{5-3\delta}{2(\delta-1)} \right) \]

or:

\[ \dot{M} = 4\pi \rho_0^2 \left( \frac{\phi_0^2}{\phi^2} \right) \left( \frac{\phi}{\phi_0} \right)^2 \text{, where } \phi = \left( \frac{3}{4} \right)^{\frac{\phi}{\phi_0}} \left( \frac{5-3\delta}{2(\delta-1)} \right) \]

\[ \rho_0 = 1.12 \text{ for } \delta = 1 \]

\[ \rho_0 = 0.25 \text{ for } \delta = 5/3 \]
Asymptotic regimes for transonic solution

$r \gg r_{\text{crit}}$:
\[ p = p_\infty, \quad T \approx T_\infty \]
\[ u \propto \frac{c_s(x)}{c_s(\infty)} \left[ \frac{GM}{c_s(x)} \right]^{3/2} \propto \frac{1}{r^2} \]

$r \ll r_{\text{crit}}$:
\[ u \approx \sqrt{\frac{GM}{r}} \] (free-fall speed)
pressure becomes negligible

\[ p \propto \frac{G M}{r^2}, \quad p_\infty \left[ \frac{GM}{c_s^2(\infty)} \right]^{3/2} \propto r^{-3/2} \times r^{-3/2} \]
\[ \rho \propto \frac{M}{r^3}, \quad \rho_\infty \left[ \frac{GM}{c_s^2(\infty)} \right]^{3/2} \propto r^{-3/2} \times r^{-3/2} \]
\[ T \propto T_\infty \left[ \frac{G M}{c_s^2(\infty)} \right]^{3/2} \propto r^{-3/2} \times r^{-3/2} \]
\[ c_s^2 \propto 1 \times T \]
\[ \frac{c_s^2}{c_s^2(\infty)} \propto 1 \]

Bondi Radius $R_B = \frac{GM}{c_s^2}$

accretion Radius $R_A = \frac{2GM}{c_s^2}$

Sonic radius $d_s = \frac{5 - 3 \delta}{4} R_B$

\[ T_\infty < c_s = \text{no shock} \]
\[ T_\infty > c_s = \text{Bondi-type accretion, but density contours are displaced} \]
\[ T_\infty \approx c_s = \text{shock: large } \delta \text{ - bow shock} \]
\[ T_\infty > c_s = \text{shock: small } \delta \text{ - tail shock} \]