Boltzman Equation: II
Masses of spherical systems: application of Jeans' equations

System to study: stationary spherically symmetric

Spherical symmetry \( \Rightarrow \frac{\partial}{\partial \phi} = 0, \frac{\partial}{\partial \theta} = 0 \)

Stationary system \( \Rightarrow \frac{\partial \rho}{\partial t} = 0 \)

Because of the symmetry bulk velocities are equal to zero: \( \langle v_\phi \rangle = \langle v_\theta \rangle = \langle v_r \rangle = 0 \)

Continuity equation \( \frac{\partial \rho}{\partial t} + \nabla (\rho \overline{v^2}) = 0 \)

is satisfied.

The Euler equation gives only one equation:

\[ (*) \]

\[ \frac{2}{n} \left( n \overline{v_r^2} \right) + \frac{n}{r} \left[ 2 \overline{v_r^2} - \left( \overline{v_\phi^2} + \overline{v_\theta^2} \right) \right] = -n \frac{\partial \rho}{\partial r} \]

where \( \phi \) is gravitational potential and \( \overline{v_r^2} \) are velocity dispersions. Because of the spherical symmetry:

\[ \langle v_\phi^2 \rangle = \langle v_\theta^2 \rangle \]

\[ \langle v_r^2 \rangle = \overline{v_r^2} \]

We introduce velocity anisotropy \( \beta(r) = 1 - \frac{\langle v_\phi^2 \rangle}{\langle v_r^2 \rangle} \)

Eq \((*)\) can be rewritten in the form:

\[ \frac{1}{n} \frac{2}{n} \left( n \overline{v_r^2} \right) + \frac{2 \beta \overline{v_r^2}}{r} = -\frac{\nabla W(r)}{r^2} \]
By rearranging terms:

\[ M(r) = -\frac{r \sqrt{V_r^2}}{G} \left\{ \frac{\partial \ln n}{\partial \ln r} + \frac{\partial \ln \sqrt{V_r^2}}{\partial \ln r} + 2\beta \right\} \]

This equation can be used to determine mass of spherical systems. It has three quantities which can be estimated using observational properties of astronomical systems.

- \( n(r) \rightarrow \) reconstructed from surface density of objects \( I(r) \)
- \( V_r^2 \rightarrow \) from line-of-sight rms velocities
- \( \beta \rightarrow \) from distribution of 1-0-3 velocities
Asymmetric drift: Rotation of a thin stellar disk

Stars rotate around center with velocity $\overline{V}_R$. They also have small chaotic velocities $\overline{V}_\theta^2$, $\overline{V}_\phi^2$, and $\overline{V}_R^2$.

In cylindrical coordinates one of Jeans' equations can be written as:

\[ \frac{2}{n} \frac{\partial}{\partial \rho} (n \overline{V}_R^2) + \frac{\partial}{\partial \rho} \left( \frac{2}{n} \rho \overline{V}_\theta \overline{V}_\phi \frac{\partial}{\partial \rho} \right) + n \left( \frac{\overline{V}_R^2 - \overline{V}_\theta^2 - \overline{V}_\phi^2}{R} + \frac{\partial \Phi}{\partial \rho} \right) = 0 \]

Here terms give: $\overline{V}_R^2 = \frac{1}{n} \int d^3 \rho \cdot \overline{V}_R^2$

Note that there is difference between $\overline{V}_R^2$ and $\overline{V}_\phi^2$:

\[ \overline{V}_\phi^2 = (\overline{V}_\phi - \overline{V}_\theta)^2 = \overline{V}_\phi^2 - \overline{V}_\theta^2 \]

$\overline{V}_\phi^2$ is azimuthal velocity dispersion.

For a stationary system and close to the galactic plane $\frac{\partial}{\delta \phi} = 0$, $\frac{\partial}{\delta z} = 0$, we have (\ref{eq:101})

\[ \frac{R}{n} \frac{\partial}{\partial R} \left( \frac{2}{n} \rho \overline{V}_\theta \overline{V}_\phi \right) + \overline{V}_R \frac{\partial}{\partial \rho} \overline{V}_\phi + \overline{V}_R^2 + R \frac{\partial \Phi}{\partial R} = 0 \]

Term $\overline{V}_\theta \overline{V}_\phi$ is small and we neglect it.

$\overline{V}_R = \sqrt{V_c^2} = (\text{circular velocity})^2$

Eq (\ref{eq:101}) takes form

\[ \overline{V}_\phi = \overline{V}_c^2 + \overline{V}_R^2 - \overline{V}_\theta^2 + \frac{R}{n} \frac{\partial}{\partial R} \left( \frac{2}{n} \rho \overline{V}_\theta \overline{V}_\phi \right) \]
For solar neighborhood $V_r^2$ is slightly larger than $6\gamma^2$, but the difference is not large. The main effect is related with the derivative of "pressure" $\frac{\partial^2}{\partial r^2}(n \nu_r^2)$. Because $n \nu_r^2$ declines with increasing radius, this derivative is negative and

\[ \frac{\nu_r^2}{\nabla} < \gamma_r^2. \]

Thus, disk rotates slower than $V_r^2 \approx \frac{GM(\odot)}{r}$

The larger $V_r^2$, the stronger is the effect.

For an exponential disk with constant height

\[ n(R, z=0) = n_0 e^{-R/\alpha} \]

\[ g_2 = \pi G \Sigma(\alpha) \]

\[ \overline{V}_{r^2} = V_0^2 e^{-R/\alpha} \]

\[ \gamma_r^2 \approx 0.5 \overline{V}_r^2 \]

\[ \frac{V_0^2}{\nabla} = V_c^2 - 6\gamma_r^2 + \overline{V}_r^2 = \frac{2R}{\alpha} \overline{V}_r^2 \]